(5) Integration by Parts :

تفاضل حاصل ضرب دالتين

We know that : $[u_{(x)} \cdot v_{(x)}]' = u_{(x)} \cdot v'_{(x)} + u'_{(x)} \cdot v_{(x)}$

Then $u_{(x)} \cdot v'_{(x)} = [u_{(x)} \cdot v_{(x)}]' - u'_{(x)} \cdot v_{(x)}$, Then

$$\int u_{(x)} \cdot v'_{(x)}\, dx = \int [u_{(x)} \cdot v_{(x)}]'\, dx - \int u'_{(x)} \cdot v_{(x)}\, dx$$

(or)

$$\boxed{\int (u \cdot v')\, dx = u \cdot v - \int (u' \cdot v)\, dx}$$

طبعاً الـ u والـ v دوال في x

لو كان الـ integrand دالتين مضروبين في بعض وكان نقدر ناخد الدالة الأكثر تعقيداً لتصبح 'v عشان تبقى سهلة أما نكاملها وناخد الـ u بالدالة اللى تفاضلها يدينا دالة أبسط من الـ u نفسها .. في هذه الحالة نقدر نستخدم طريقة الـ integration by parts في الحل .

لأن الـ integrand في هذه الطريقة قسمناه لجزئين سميت هذه الطريقة من إيجاد التكامل بالـ integration by parts .

خد بالك 😊 إنك بتبص لاختيار الـ 'v أولاً إذا لم تستطع بتختار الـ u وطبيعي الجزء الآخر هيكون الـ u أو الـ 'v ع الترتيب .

ملحوظة : بنحط ثابت التكامل في آخر خطوة في الحل ، يعني مش لازم نحط ثابت أما نجيب الـ v .

Find: (i) $I = \int x . e^x \, dx$ (ii) $J = \int x^2 e^x \, dx$

Ans:

(i) take $u = x$, $v' = e^x$, Then

$u' = 1$, $v = e^x$, so

$$I = u.v - \int u' v \, dx = x.e^x - \int e^x \, dx$$

$$\therefore \boxed{I = x.e^x - e^x + C}$$

$$\int x . e^x \, dx = \int \underbrace{x}_{u} \, d(\underbrace{e^x}_{v}) = x.e^x - \int e^x dx = x e^x - e^x + C$$

(ii) take $u = x^2$, $v' = e^x$, Then

$u' = 2x$, $v = e^x$, so

$$J = u.v - \int u' v \, dx = x^2 e^x - 2\int x . e^x \, dx$$

دى تعملها زى المسألة السابقة

Then, $J = x^2 e^x - 2\left[x e^x - e^x\right]$

$$\boxed{J = e^x (x^2 - 2x - 1) + B}$$

Note: $\int u . v' = u.v - \int u' . v . dx$ can be written as

$$\int u . dv = u.v - \int v \, du$$ كده بتطبيق القاعدة اللى فوق

ملحوظة: فى حالة اختيارك لل u و ال v ولقيت إن الحل بيتعقد .. أعد الحل مع تبديل ال u وال v عكس اللى كنت اختارته.

Find: $I = \int x \sin x \, dx$

Ans: take $u = x$, $v' = \sin x$, Then

$u' = 1$, $v = -\cos x$, so

$$I = u v - \int u' v \, dx = -x \cos x - \int -\cos x \, dx$$

Then $I = \boxed{\sin x - x \cos x + C}$ #

Find: $I = \int \ln x \, dx$

Ans: take $u = \ln x$, $V' = 1$, Then

$u' = \frac{1}{x}$, $V = x$, so

$$I = uV - \int u'V\,dx \Rightarrow I = x.\ln x - \int dx$$

$$\therefore I = \boxed{x.\ln x - x + C} \;\#$$

لاحظ: في مسائل تكامل الـ ln
يتم اختيار الـ u بدالة الـ ln
وذلك لصعوبة تكاملها إذا
وُضعت الـ V' بـ ln

Find: $J = \int x^2 \ln x \, dx$

Ans: take $u = \ln x$, $V' = x^2$, Then $u' = \frac{1}{x}$, $V = \frac{1}{3}x^3$

$$J = uV - \int u'V\,dx = \frac{1}{3}x^3.\ln x - \int \frac{1}{3}x^2\,dx$$

$$J = \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C} \;\#$$

Find: $K = \int \sin^{-1}x \, dx$

Ans: take $u = \sin^{-1}x$, $V' = 1$, Then

$u' = \frac{1}{\sqrt{1-x^2}}$, $V = x$, so

$$K = uV - \int u'V\,dx = x.\sin^{-1}x + \int \frac{-x}{\sqrt{1-x^2}}\,dx \longrightarrow (1)$$

$$J = \int \frac{-x}{\sqrt{1-x^2}}\,dx \Rightarrow \text{Put } u = \sqrt{1-x^2} \Rightarrow u^2 = 1 - x^2 \Rightarrow \cancel{2}u\,du = -\cancel{2}x\,dx$$

$$u\,du = -x\,dx \text{ . Then}$$

$$J = \int \frac{\cancel{u}.du}{\cancel{u}} = u + C = \sqrt{1-x^2} + C \rightarrow \text{sub. in } (1) \text{ . Then}$$

$$K = \boxed{x.\sin^{-1}x + \sqrt{1-x^2} + C} \;\#$$

When the integrand contains inverse trigonometric or hyperbolic functions use the technique of integration by parts, where take $\underline{u}$ to equal the inverse $f\underline{n}$.

Find:

1. $I = \int \cos^{-1}x \, dx$.
2. $J = \int \tan^{-1}x \, dx$.
3. $K = \int \sec^{-1}x \, dx$.
4. $L = \int \csc^{-1}x \, dx$.
5. $M = \int \cot^{-1}x \, dx$.

Ans:

1. take $u = \cos^{-1}x$, $V' = 1$, Then

$u' = \frac{-1}{\sqrt{1-x^2}}$, $V = x$, so

$$I = u.V - \int u'.V \, dx = x \cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \cos^{-1}x - \int \frac{1}{2}(1-x^2)^{\frac{-1}{2}}(-2x)\, dx = \boxed{x.\cos^{-1}x - (1-x^2)^{\frac{1}{2}} + C} \#$$

2. take $u = \tan^{-1}x$, $V' = 1$, Then

$u' = \frac{1}{1+x^2}$, $V = x$, so

$$J = u.V - \int u'V \, dx = x.\tan^{-1}x - \int \frac{x}{1+x^2} dx = x \tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$= \boxed{x.\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C}$$

3. take $u = \sec^{-1}x$, $V' = 1$, Then

$u' = \frac{1}{x\sqrt{x^2-1}}$, $V = x$, so

$$K = u.V - \int u V' \, dx = x \sec^{-1}x - \int \frac{1}{\sqrt{x^2-1}} dx$$

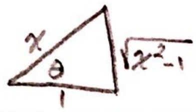
$$= \boxed{x \sec^{-1}x - \ln|x + \sqrt{x^2-1}| + C}$$

Put $x = \sec\theta \Rightarrow dx = \sec\theta \tan\theta \, d\theta$

Then: $\int \frac{\sec\theta \cancel{\tan\theta}}{\cancel{\tan\theta}} d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln|x + \sqrt{x^2-1}| + C$



4. take $u = \csc^{-1}x$, $V' = 1$, Then

$u' = \frac{-1}{x.\sqrt{x^2-1}}$, $V = x$, so

$$L = u.V - \int u'V \, dx = x.\csc^{-1}x + \int \frac{x}{x\sqrt{x^2-1}} dx$$

$$L = \boxed{x.\csc^{-1}x + \ln|x + \sqrt{x^2-1}| + C} \#$$

5. take $u = \cot^{-1}x$, $V' = 1$, Then $u' = \frac{-1}{1+x^2}$, $V = x$, Then

$$M = u.V - \int u'V \, dx = x.\cot^{-1}x + \int \frac{x}{1+x^2} dx = x \cot^{-1}x + \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

Then $M = \boxed{x \cot^{-1}x + \frac{1}{2}\ln(1+x^2) + C} \#$

Find : Integration of inverse hyperbolic functions.

1. $I = \int \sinh^{-1} x \, dx$.
2. $J = \int \cosh^{-1} x \, dx$.
3. $K = \int \tanh^{-1} x \, dx$.
4. $L = \int \operatorname{sech}^{-1} x \, dx$.
5. $M = \int \operatorname{csch}^{-1} x \, dx$.
6. $N = \int \coth^{-1} x \, dx$.

Integration By Parts

$UV - \int U'V \, dx$

Ans :

1. take $U = \sinh^{-1} x$, $V' = 1$, Then

$U' = \frac{1}{\sqrt{1+x^2}}$, $V = x$, So

$$I = UV - \int U'V \, dx = x.\sinh^{-1} x - \int \frac{x}{\sqrt{1+x^2}} dx = x.\sinh^{-1} x - \int \frac{1}{2}(1+x^2)^{-\frac{1}{2}} (2x) \, dx$$

$$I = \boxed{x \sinh^{-1} x - \sqrt{1+x^2} + C} \quad \#$$

2. take $U = \cosh^{-1} x$, $V' = 1 \Rightarrow U' = \frac{1}{\sqrt{x^2-1}}$. $V = x$, So

$$J = UV - \int U'V \, dx = x \cosh^{-1} x - \int \tfrac{1}{2}(2x)(x^2-1)^{-\frac{1}{2}} dx = \boxed{x \cosh^{-1} x - \sqrt{x^2-1} + C} \quad \#$$

3. take $U = \tanh^{-1} x$, $V' = 1 \Rightarrow U' = \frac{1}{1-x^2}$, $V = x$, Then

$$K = x \tanh^{-1} x - \int \frac{x}{1-x^2} dx = x \tanh^{-1} x + \frac{1}{2}\int \frac{-2x}{1-x^2} dx = \boxed{x \tanh^{-1} x + \frac{1}{2} \ln|1-x^2| + C}$$

4. take $U = \operatorname{sech}^{-1} x$, $V' = 1 \Rightarrow U' = \frac{-1}{x\sqrt{1-x^2}}$, $V = x$, Then

$$L = x \operatorname{sech}^{-1} x + \int \frac{dx}{\sqrt{1-x^2}} = \boxed{x.\operatorname{sech}^{-1} x + \sin^{-1} x + C}$$

5. take $U = \operatorname{csch}^{-1} x$, $V' = 1$, Then $U' = \frac{-1}{x.\sqrt{1+x^2}}$. $V = x$, So

$$M = x \operatorname{csch}^{-1} x + \int \frac{dx}{\sqrt{1+x^2}} = \boxed{x \operatorname{csch}^{-1} x + \sinh^{-1} x + C}$$

6. take $U = \coth^{-1} x$, $V' = 1 \Rightarrow U' = \frac{1}{1-x^2}$, $V = x$, Then

$$N = x \coth^{-1} x + \frac{1}{2}\int \frac{-2x}{1-x^2} dx = \boxed{x \coth^{-1} x + \frac{1}{2} \ln|1-x^2| + C}$$

Find: $I = \int x^2 \sin x \, dx$

Ans:

Take: $u = x^2$, $V' = \sin x \Rightarrow u' = 2x$, $V = -\cos x$, Then

$$I = uV - \int u'V \, dx = -x^2 \cos x + 2 \underbrace{\int x \cos x \, dx}_{J}$$

خد بالك

لما تطبق ال (I.B.P) مرتين في نفس المسألة -- يُراعى إنك تاخد نفس الاختيارات لأنك لو بدلت هترجع للمسألة الأصلية.

Integration By Part,

take $u = x$, $V' = \cos x$

$u' = 1$, $V = \sin x$

$J = x \sin x - \int \sin x \, dx$

$= x \sin x + \cos x + C$, Then

خد بالك

إن كلا الجزئين سهلين في التكامل إلا إننا فضلنا اختيار u بـ x لأننا أسهل في تفاضلها من ال sin x.

$$I = \boxed{-x^2 \cos x + 2x \sin x + 2 \cos x + C}$$

Find: $I = \int e^x \cos x \, dx$.

Ans: take $u = e^x$, $V' = \cos x$, Then

$u' = e^x$, $V = \sin x$, So

$$I = uV - \int u'V \, dx = e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{J}$$, Then

هنعملها برده I.B.P مع نأخذ ال u وال V' بنفس اختيارات الفرضة الأولى

$u = e^x$, $V' = \sin x$

$u' = e^x$, $V = -\cos x$

$$J = -e^x \cos x + \underbrace{\int e^x \cos x \, dx}_{I}$$

$J = -e^x \cos x + I$, Then

$$I = e^x \sin x + e^x \cos x - I \Rightarrow 2I = e^x [\sin x + \cos x]$$, Then

$$\boxed{I = \frac{1}{2} e^x [\sin x + \cos x] + C}$$

Find: $I = \int e^x \sin x \, dx$

في هذا النوع من المسائل.. ممكن اختيار الـ u والـ v' حسب مزاجك لكن يجب أن تلتزم بهذا الاختيار لاحقاً عند ظهور الـ I.B.P فيما بعد.

Ans: take $u = \sin x$, $v' = e^x$, Then

$u' = \cos x$, $v = e^x$, so

$$I = e^x \sin x - \int e^x \cos x \, dx \longrightarrow \underline{\text{I.B.P}}, \text{ called } (J)$$

$u = \cos x$, $v' = e^x$

$u' = -\sin x$, $v = e^x$

$$\Bigg\} \; J = e^x \cos x + \underbrace{\int e^x \sin x \, dx}_{I} , \text{ Then}$$

$$J = e^x \cos x + I , \text{ so}$$

$$I = e^x \sin x - e^x \cos x - I \Rightarrow \boxed{I = \tfrac{1}{2} e^x [\sin x - \cos x] + C}$$

Find: $I = \int \sec^3 x \, dx$

Ans: $I = \int \sec x \cdot \sec^2 x \, dx$

هنا $\sec^2 x$ أسهل عند تكاملها عشان كده هنا خد $v' = \sec^2 x$

take $u = \sec x$, $v' = \sec^2 x$, Then

$u' = \sec x \tan x$, $v = \tan x$, so

$$I = \sec x \cdot \tan x - \int \sec x \cdot \tan^2 x \, dx$$

$$= \sec x \cdot \tan x - \int \sec x \, [\sec^2 x - 1] \, dx$$

$$= \sec x \cdot \tan x - \underbrace{\int \sec^3 x \, dx}_{I} + \int \sec x \, dx$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\boxed{I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C}$$

مُجَدول — التكامل بالتجزيء المتتابع

Successive integration by Parts: (tabular integration)

→ The usual version of integration by Parts in general can be shown by a tabular integration with only two rows as shown:

$$\begin{array}{ccc} u_{(x)} & & V'_{(x)} \\ & \searrow^{+} & \\ u'_{(x)} & \xrightarrow{-\int} & V_{(x)} \end{array}$$

$$\boxed{I = u_{(x)} \cdot V_{(x)} - \int u'_{(x)} V_{(x)}\, dx}$$

→ successive integration by Parts is useful technique for Performing integration by Parts multiple times in a row.

→ The basic technique is to split the integrand into two Pieces, (تكرارياً) iteratively differentiate one Part and integrate the other and arrange the result into a table.

→ for example, $I = \int x^2 e^{2x}\, dx$, The integrand $x^2 \cdot e^{2x}$ can be splited into two Parts x^2 and e^{2x}. Then

	diff. ↓	integrate ↓
	x^2	e^{2x}
	$2x$	$\frac{1}{2} e^{2x}$
	2	$\frac{1}{4} e^{2x}$
	0	$\frac{1}{8} e^{2x}$

To obtain the indefinite integral from this table, draw slanted lines with alternating signs (started with +ve sign), take the Product of the terms at the end of each arrow with the sign on the arrow, and sum

$$\begin{array}{ccc} x^2 & & e^{2x} \\ & \searrow^{+} & \\ 2x & & \frac{1}{2} e^{2x} \\ & \searrow^{-} & \\ 2 & & \frac{1}{4} e^{2x} \\ & \searrow^{+} & \\ 0 & & \frac{1}{8} e^{2x} \end{array}$$

, Then

$$I = x^2 * \frac{1}{2} e^{2x} - 2x * \frac{1}{4} e^{2x} + 2 * \frac{1}{8} e^{2x} + C = \boxed{\frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C}$$

- This method will always work when the left column eventually becomes 0, i.e. when the top of the left column is a polynomial.
- However, with a slight modification, the technique of tabular integration can be useful even when the left column dosn't eventually becomes 0. The modification is to add a horizontal line on the bottom row, give it a sign (in the same alternating fashion), and include the integral of the product of these terms (with the sign) in the sum.
- for example. $I = \int e^{2x} \sin 3x \, dx$, Then

	sign	
e^{2x}		$\sin 3x$
$2e^{2x}$	$+$	$\frac{-1}{3}\cos 3x$
$4e^{2x}$	$-$	$\frac{-1}{9}\sin 3x$
	$+\int$	

$$\int e^{2x}\cdot \sin(3x)\,dx = \frac{-1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\cdot\sin 3x - \frac{4}{9}\int e^{2x}\sin(3x)\,dx$$

$$\frac{13}{9}\int e^{2x}\sin(3x)\,dx = \frac{-1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$

$$\int e^{2x}\sin(3x)\,dx = \frac{-3}{13}e^{2x}\cos 3x + \frac{2}{13}e^{2x}\sin 3x + C$$

$$= \boxed{\frac{e^{2x}}{13}\left(2\sin 3x - 3\cos 3x\right) + C}$$

خد بالك

> The reason a horizontal line with an integral isn't necessary when the left column becomes 0 is quite simple: the integral is still there, but it's equal to zero.

Find: (i) $I = \int x^4 . e^{-x}\, dx$. (ii) $J = \int x^4 . e^{x}\, dx$.

(iii) $K = \int x^3 \sin x\, dx$. (iv) $L = \int x^3 \cos x\, dx$.

Ans:

(i) split the integrand to,

$$\begin{array}{ccc} x^4 & & e^{-x} \\ 4x^3 & + & -e^{-x} \\ 12x^2 & - & e^{-x} \\ 24x & + & -e^{-x} \\ 24 & - & e^{-x} \\ 0 & + & -e^{-x} \end{array}$$

$$I = -x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24x . e^{-x} - 24 e^{-x} + C$$

$$I = \boxed{-e^{-x}\left[x^4 + 4x^3 + 12x^2 + 24x + 24\right] + C}$$

(ii) split the integrand to,

$$\left.\begin{array}{ccc} x^4 & & e^{x} \\ 4x^3 & + & e^{x} \\ 12x^2 & - & e^{x} \\ 24x & + & e^{x} \\ 24 & - & e^{x} \\ 0 & + & e^{x} \end{array}\right\} \quad J = \boxed{e^{x}\left[x^4 - 4x^3 + 12x^2 - 24x + 24\right] + C}$$

(iii) split the integrand as follow,

$$\begin{array}{lcl} x^3 & & \sin x \\ 3x^2 & \xrightarrow{+} & -\cos x \\ 6x & \xrightarrow{-} & -\sin x \\ 6 & \xrightarrow{+} & \cos x \\ 0 & \xrightarrow{-} & \sin x \end{array}$$

$$K = \boxed{-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C}$$

(iv) split the integrand as follow,

$$\begin{array}{lcl} x^3 & & \cos x \\ 3x^2 & \xrightarrow{+} & \sin x \\ 6x & \xrightarrow{-} & -\cos x \\ 6 & \xrightarrow{+} & -\sin x \\ 0 & \xrightarrow{-} & \cos x \end{array}$$

$$L = \boxed{x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C}$$